On Tractability Aspects of Optimal Resource Allocation in OFDMA Systems

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Abstract

Joint channel and rate allocation with power minimization in orthogonal frequency-division multiple access (OFDMA) has attracted extensive attention. Most of the research has dealt with the development of sub-optimal but low-complexity algorithms. In this paper, the contributions comprise new insights from revisiting tractability aspects of computing optimum. Previous complexity analyses have been limited by assumptions of fixed power on each subcarrier, or power-rate functions that locally grow arbitrarily fast. The analysis under the former assumption does not generalize to problem tractability with variable power, whereas the latter assumption prohibits the result from being applicable to well-behaved power-rate functions. As the first contribution, we overcome the previous limitations by rigorously proving the problem's NP-hardness for the representative logarithmic rate function. Next, we extend the proof to reach a much stronger result, namely that the problem remains NP-hard, even if the channels allocated to each user is restricted to a consecutive block with given size. We also prove that, under these restrictions, there is a special case with polynomial-time tractability. Then, we treat the problem class where the channels can be partitioned into an arbitrarily large but constant number of groups, each having uniform gain for every individual user. For this problem class, we present a polynomial-time algorithm and prove optimality guarantee. In addition, we prove that the recognition of this class is polynomial-time solvable.

Keywords: orthogonal frequency-division multiple access, resource allocation, tractability.

1 Introduction

In orthogonal frequency division multiple access (OFDMA) systems, resource allocation amounts to finding the optimal assignment of subcarriers to users, and, for each user, the allocation of rate or power over the assigned subset of sub-carriers. In this paper, we focus on the problem of minimizing the total transmit power, subject to delivering specified data rates, by power assignment and subcarrier allocation. The popularity of OFDMA for wireless communications has led to an intense research effort in this area. A majority of the works has focused on heuristic and thus sub-optimal solutions, see, for example, [1–7], and the references therein. For global optimality, a branch-and-bound algorithm is developed in [8].

We address a complementary but fundamental aspect of the resource allocation problem: To what extent is it tractable? In contrast to the significant amount of research on algorithms, works along the line of tractability analysis are few [9, 10]. The edge of fundamental understanding of problem tractability is formed in respect of the following limitations. In [10], the power on each subcarrier is assumed to be given, i.e., it can be arbitrarily set to any fixed value for the purpose of proving complexity. Thus the result does not apply to the problem where powers are optimization variables. In [9], the resource allocation problem is shown to be NP-hard for one particular type of rate function $r_{mn}(P) \in \mathcal{C}_{\text{inc}}$, where $r_{mn}(P)$ is the rate as a function of power P for the mth user and nth subcarrier, and $\mathcal{C}_{\text{inc}}$ is the set of all increasing functions such that $r_{mn}(0) = 0$. By this result, there exists some (but possibly ill-behaved[1]) function in $\mathcal{C}_{\text{inc}}$ for which the problem is NP-hard. However, the result does not carry over to well-behaved subclasses of functions in $\mathcal{C}_{\text{inc}}$. In fact, for any $r_{mn}(P) \in \mathcal{C}_{\text{linear}}$, where $\mathcal{C}_{\text{linear}}$ is the set of linear functions, the problem is solvable in polynomial time (see Section 3). Thus, whereas dealing with $\mathcal{C}_{\text{inc}}$ is in general intractable for the problem in question, there exists some subclass in $\mathcal{C}_{\text{inc}}$ that admits global optimality at low complexity.

For the (wide) class of functions in $\mathcal{C}_{\text{inc}}$ but not in $\mathcal{C}_{\text{linear}}$, the tractability of the resource al-

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[1]Indeed, the proof in [9] relies on a power function (i.e., the inverse of the rate function) growing arbitrarily fast for arbitrarily small rate increase, meaning that the function is not locally Lipschitz continuous.

location problem remains unknown and thus calls for investigation. The aspect is of most relevance for the class of increasing and concave functions $\mathcal{C}_{\text{concave}}$, for which $\mathcal{C}_{\text{linear}} \subset \mathcal{C}_{\text{concave}} \subset \mathcal{C}_{\text{inc}}$ holds. Indeed, it is very commonly assumed that the rate function is given by $r_{mn}(P) = \log(1 + g_{mn}P)$, with $g_{mn} > 0$. We will thus investigate the complexity for this representative case of $\mathcal{C}_{\text{concave}}$, and, if the problem is NP-hard for the function, examine to what extent restrictions on input structure (e.g., assuming the number of subcarrier for each user is part of the input) will admit better tractability. The specific contributions are as follows.

- We rigorously prove that given the representative logarithmic rate function $r_{mn}(P) = \log(1 + g_{mn}P)$ with $g_{mn} > 0$, the resource allocation problem is NP-hard. Since this rate function is in $\mathcal{C}_{\text{concave}}$, it follows that in the hierarchy $\mathcal{C}_{\text{linear}} \subset \mathcal{C}_{\text{concave}} \subset \mathcal{C}_{\text{inc}}$, the hardness result holds except for the linear case. The contribution leads a significant refinement of the result of tractability.

- We extend the NP-hardness analysis to arrive at a much stronger result. Namely, the problem remains NP-hard, even if the following two restrictions are *jointly* imposed: 1) the subcarriers allocated to each user form a consecutive block, and 2) the block size is given as part of the problem's input. We also prove that, with these two restrictions, there is a special case admitting polynomial-time tractability.

- We identify a tractable problem subclass, with the structure that the channels can be partitioned into an *arbitrarily many but constant* number of groups, possibly with varying group size, and the channels within each group have uniform gain for each user (but may differ by user). The original problem is, in fact, equivalent to having the number of groups equal to the number of subcarriers. The tractable subclass goes beyond the logarithmic rate function – the result holds as long as the single-user resource allocation is tractable. Moreover, we prove that recognizing this problem class is tractable as well.

The remainder of the paper is organized as follows. The system model is given in Section 2. In Section 3, we provide and prove the base result of NP-hardness. Section 4 is devoted to

the problem's tractability with restrictions on channel allocation. In Section 5, we consider the problem class with structured channel gain and prove its tractability. In addition, we prove that recognizing this problem class is computable in polynomial time. Conclusions are given in Section 6.

2 System Model

Consider an OFDMA system with M users and N subcarriers. In this paper, the terms subcarrier and channel are used interchangeably. For convenience, we define sets $\mathcal{M} = \{1, \ldots, M\}$ and $\mathcal{N} = \{1, \ldots, N\}$. Notation $r_{mn}(P_{mn})$ is reserved for the rate as a function of the transmission power $P_{mn} \geq 0$ for the mth user and nth subcarrier. The inverse function r_{mn}^{-1}, returning the power for supporting a given rate, is denoted by f_{mn}. The required rate of user m is denoted by R_m.

The forthcoming analysis focuses on the representative increasing, concave rate function $r_{mn}(P_{mn}) = \log_2(1 + g_{mn}P_{mn})$, where $g_{mn} > 0$ represents the channel gain (normalized such that the noise variance is one). Hence, the corresponding signal-to-noise ratio (SNR) is $g_{mn}P_{mn}$. Some of the tractability results generalize to $\mathcal{C}_{\text{inc}}$; we will explicitly mention the generalization when it applies.

The optimization problem is to minimize the total power by joint channel and rate allocation, such that the users' required rate targets are met. The problem is formulated formally below. Throughout the rest of the paper, we refer to the problem as minimum-power channel allocation (MPCA).

Input: User set $\mathcal{M} = \{1, \ldots, M\}$ and channel set $\mathcal{N} = \{1, \ldots, N\}$ with $M \leq N$, positive channel gain $g_{mn}, m \in \mathcal{M}, n \in \mathcal{N}$, and positive rate targets $R_m, m \in \mathcal{M}$.

Output: A channel partitioning represented by $(\mathcal{N}_1, \ldots, \mathcal{N}_M)$, where $\mathcal{N}_m \subset \mathcal{N}, m \in \mathcal{M}$, $\mathcal{N}_{m_1} \cap \mathcal{N}_{m_2} = \emptyset$, for all $m_1, m_2 \in \mathcal{M}, m_1 \neq m_2$, and non-negative power $P_{mn}, m \in$

$\mathcal{M}, n \in \mathcal{N}_m$, such that $\sum_{n \in \mathcal{N}_m} \log_2(1 + P_{mn}g_{mn}) \geq R_m, m \in \mathcal{M}$ and the total power $\sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}_m} P_{mn}$ is minimized.

By the rate function, there is a unique mapping between rate allocation and power expenditure. In addition, at optimum, the rate may be zero on some of the allocated channels of a user. Given the channel allocation, power optimization is determined by solving the single-user rate allocation problem by water-filling [11] in linear time (e.g., [7]). The combinatorial nature of the multi-user problem stems from the fact that the users can not share a subcarrier, and the core of problem-solving is channel partitioning.

3 Tractability of MPCA: Base Results

To motivate the investigation of tractability in view of the current literature, two remarks are noteworthy. First, an NP-hard problem may become tractable by imposing restrictions to the structure of its input parameters (e.g., the shape of the objective function). An example is the traveling salesman problem (TSP) having a cost function with the so called Klyaus-matrix structure (meaning that inversed triangular distance inequality holds). In this case, TSP is solvable in polynomial time [12]. For MPCA, it is in fact polynomially solvable, if the power-rate function would be linear (see the end of this section). As the second remark, an NP-hard problem may become tractable by removing some of its constraints (and hence enlarging the solution space). For example, minimum spanning tree (MST) with constrained node degree is NP-hard, but becomes easily-solved if this constraint is removed.

It has been widely accepted that MPCA and other related OFDMA resource allocation problem are difficult. However, to the best of our knowledge, no formal analysis other than the results in [9, 10] is available. The study in [10] formalizes the NP-hardness result with the assumption that the powers on all channels are given. This is equivalent to introducing constraints fixing the power values. By the second remark above, the result does not answer the tractability if these constraints are removed (that is, the original problem with variable power). Indeed, for

any linear rate function as well as the problem class in Section 5, the NP-hardness proof in [10] remains valid for fixed power, but these problem classes with variable power are solvable in polynomial time. For the analysis in [9], the proof requires unbounded power growth for arbitrarily small rate increase. Specifically, for the power function f, $f(n + \frac{1}{k})$ is k times higher than $f(n)$, for arbitrary positive integers n and k. This assumption of ill-behaved power-rate function excludes not only $f(x) = 2^x - 1$ (the inverse of the logarithmic rate function), but also all locally Lipschitz continuous functions. Recall that a (not necessarily continuous) function f is locally Lipschitz continuous, if for any x, there exists a small real number ϵ and an arbitrarily large but constant real number D, such that $|f(x + \epsilon) - f(x)| \leq D\epsilon$, that is, the growth of the function is bounded when the change in the input diminishes. Clearly, f is not locally Lipschitz continuous, if $f(n + \frac{1}{k})$ increases by factor k over $f(n)$ for any n and arbitrarily large k. Hence, by the previous remark of the impact of cost function on tractability, the tractability under more well-behaved functions calls for investigation.

We provide the tractability results that overcome the limitations of the currently available analysis in two aspects. First, we present a rigorous proof of the problem's NP-hardness with the representative logarithmic rate function. Second, in the next section, we extend the proof to reach a much stronger result, stating that the problem remains NP-hard even with two heavy restrictions on channel allocation.

Theorem 1. *MPCA, as defined in Section 2, is NP-hard.*

Proof. As the proof is rather technical, we outline the basic idea and defer the details to Appendix A. The proof uses a reduction from 3-satisfiability (3-SAT). Two groups of users are defined. At optimum, each user in the first group either uses one channel of superior channel gain, or splits the rate on three inferior channels, but not both. This corresponds to the true/false value assignment in 3-SAT. The optimal power for this group of users is a constant, while the optimal power for the second user group gives the correct answer to 3-SAT. □

Corollary 2. *MPCA remains NP-hard, even if the rate requirements of the users are uniform.*

Proof. Follows immediately from the equal-rate values used in the proof of Theorem 1. $\square$

Earlier in this section, it was claimed that MPCA with any linear rate function is tractable. Even if this case is not much of practical interest, it is instructive in showing the importance of input assumption on problem tractability.

Theorem 3. *MPCA with linear rate function $r_{mn}(P) = \ell_{mn}P$, where $\ell_{mn} \geq 0$, $m \in \mathcal{M}, n \in \mathcal{N}$, is solvable in polynomial time.*

Proof. Since the rate (and hence power) function is linear, it follows that for any user m, it is optimal to allocate the entire rate R_m to a single channel. Specifically, denoting by $\mathcal{N}_m$ the channel set allocated to m, the optimal selection is the channel giving $\min_{n \in \mathcal{N}_m} 1/\ell_{mn}$. Because of this structure at optimum, MPCA reduces to pairing the M users with M out of the N channels. Hence the problem is equivalent to a minimum-weight matching problem (also known as minimum-cost assignment [13]) in a bipartite graph with node sets $\mathcal{M} \cup \{M + 1, \ldots, N\}$ and $\mathcal{N}$; the former represents the augmentation of $\mathcal{M}$ by $M - N$ artificial users. For edge (m, n), with $m \in \mathcal{M}$ and $n \in \mathcal{N}$, the cost is $1/\ell_{mn}$. All edges adjacent to artificial users have zero cost. Because the assignment problem is polynomial-time solvable, the theorem follows. $\square$

Remark In light of Theorem 1, Corollary 2, and Theorem 3, we remark on the significance of assumption of input on tractability, by revisiting the result provided in [10]. In this reference, the result is proven for fixed power, that is, the power of each channel is part of problem input. Under this condition, [10] provides an elegant hardness proof of a reduction from the number partitioning problem, by setting specific power values on the channels. As long as power is fixed, a line-by-line copy of the proof in [10] remains valid even if the underlying MPCA rate function is linear. A similar observation applies to the tractable problem class that will be detailed in Section 5. In conclusion, for the original MPCA problem having power allocation as part of the output (for which the key assumption of [10] does not apply), our analysis provides new insights in the tractability rather than contradicting the previous results. $\square$

4 Tractability with Restrictions on Channel Allocation

Consider imposing jointly two restrictions to channel allocation. First, the number of channels to be allocated to each user is given. Tractability under this restriction is of significance to two-phase OFDMA resource allocation (see [14]) that determines the number of channels per user in phase one, followed by channel allocation in phase two. The second restriction is the use of consecutive channels, that is, the channels of every user must be consecutive in the sequence $1, \ldots, N$; this channel-adjacency constraint has been considered in, for example, [15]. We prove that MPCA remains NP-hard even with these two seemingly strong restrictions, although there is a special case admitting polynomial-time tractability.

Theorem 4. *MPCA remains NP-hard, even if the number of channels allocated to each user, i.e., the cardinality of $\mathcal{N}_m, m = 1, \ldots, M$, is given in the input, and $\mathcal{N}_m, m = 1, \ldots, M$ must contain consecutive elements in the channel sequence $1, \ldots, N$.*

Proof. As we prove the result via an augmentation of reduction proof of in Appendix A, the details are deferred to Appendix B. In brief, the proof is built upon introducing a set of additional channels in the MPCA instance in Appendix A, in a way such that, at optimum, the channels allocated to each user are consecutive and the corresponding cardinality is known. $\square$

Consider further narrowing down the problem to the case where $\mathcal{N}$ consists of M uniform-sized subsets of consecutive channels, and N is a multiple of M. For this case, the problem is tractable, as proven below.

Theorem 5. *If N is divisible by M, and it is restricted to allocate exactly $\frac{N}{M}$ consecutive channels to each user, then MPCA is tractable with time complexity $O(\max\{M^3, MN\})$.*

Proof. We prove the result by a polynomial-time transformation to the well-solved matching problem in a bipartite graph. For the problem setting in question, channel partition is unique - the M subsets created by the partition are $\{1, \ldots, N/M\}$, $\{N/M + 1, \ldots, 2N/M\}$, $\ldots$, $\{N - N/M + 1, \ldots, N\}$, each containing N/M channels. The bipartite graph has $2M$ nodes

representing the users and the subsets of channels. For each pair of the two node groups, say user m and channel subset S, there is an edge of which the cost equals the power of meeting the user rate using the channels in S. This cost is computed in $O(\frac{N}{M})$ time (by single-user rate allocation). The complexity of computing all the edge costs is hence $O(MN)$. Maximum-weighted matching in a bipartite graph of V nodes and E edges is solved in $O(V^2 \log V + VE)$ time [13]. In our case, the second term is dominating, giving a time complexity of $O(M^3)$, which completes the proof. $\qquad\square$

Theorem 5 provides a generalization of the trivial case of $M = N$. For $M = N$, solving MPCA amounts to finding an optimal matching with complexity $O(M^3)$. Moreover, from the proof, it follows that the analysis generalizes to any rate function in $\mathcal{C}_{\text{inc}}$, except that the complexity of single-use rate allocation has to be accounted for accordingly. The observation yields the following corollary.

Corollary 6. *The time required for computing the optimum to the MPCA problem class in Theorem 5 is of $O(\max\{M^3, M^2 T(\frac{N}{M})\})$, where $T(\frac{N}{M})$ denotes the time complexity of optimal rate allocation of a single user on $\frac{N}{M}$ channels.*

5 A Tractable Problem Class

Denote by K a (possibly large) fixed positive integer independent of M or N. Let $\mathcal{K} = \{1, 2, \cdots, K\}$. Consider MPCA in which the channels can be partitioned into (at most) K groups where every user in the same group has the same channel gain, but the channel gains differ by user. Thus, we can write $\mathcal{N} = \bigcup_{k \in \mathcal{K}} \mathcal{N}_k$, such that for any channel $n \in \mathcal{N}_k$, the channel gain depends only on the user index m, i.e., $g_{mn} = g_m$. Equivalently, the rate functions belong to the subclass that satisfy $r_{mn} = r_m, n \in \mathcal{N}_k, k \in \mathcal{K}, m \in \mathcal{M}$. Note that the channel groups may vary in size, and the channel gain still differs by user within each group. In the sequel, we refer to the problem class as K-MPCA. The problem class is justified by scenarios with K distinct bands and channel difference is overwhelmingly contributed by the separation

of the bands in the spectrum, whereas the subcarriers with each band are considered invariant for each user.

Consider 1-MPCA. The problem structure is significantly simpler than the general case. Namely, the optimization decision is no longer which, but how many channels each user should use. It follows that, for any subset of users $\mathcal{M}' \subset \mathcal{M}$, the optimum allocation of h channels (with $h \geq |\mathcal{M}'|$) among the users in $\mathcal{M}'$ is independent of channel allocation of the rest the users. Thus 1-MPCA exhibits an optimal substructure, i.e., a part of the optimal solution is also optimal for that part of the problem. The observation leads to a dynamic programming line of argument for problem-solving. As proven below, the solution strategy guarantees optimality in polynomial time.

Theorem 7. *Global optimum of* 1*-MPCA can be computed by dynamic programming in* $O(MN^2)$ *time.*

Proof. Consider the partial problem of optimally allocating h channels to the users in $\{1, \ldots, m\}$ with $h \geq m$, and $c_m(h)$ the corresponding optimum power. Clearly, at the optimum of this sub-problem, the number of channels of user m is an integer in the set $\{1, 2, \ldots, h - m + 1\}$. (The upper bound $h - m + 1$ corresponds to having $m - 1$ channels left for the other $m - 1$ users.) Allocating $k \in \{1, 2, \ldots, h - m + 1\}$ channels to user m, the power equals $p_m^k + c_{m-1}(h - k)$, where p_m^k denotes the power of user m with k channels. This gives the following recursive formula for computing the optimal number of channels for user m.

$$c_m(h) = \min_{k=1,\ldots,h-m+1} \{p_m^k + c_{m-1}(h - k)\} \tag{1}$$

We arrange the values $c_m(h)$ for $m = 1, \ldots, M$ and $h = 1, \ldots, N$ in an $M \times N$ matrix. Entries corresponding to infeasible solutions are called invalid, and their values are denoted by ∞. In the matrix, $c_m(h) = \infty$ for all entries where $h < m$, or $h > N - M + m$. We compute the valid entries as follows. For the first row, computing the entries $c_1(1), \ldots, c_1(N - M + 1)$ in the given order are straightforward, and each entry requires $O(1)$ computing time. Next, entries $c_m(m)$,

10

i.e., one channel per user for the first m users, are calculated in $O(M)$ time for $m = 1, \ldots, M$. The bulk of the computation calculates the remaining entries row by row, starting from row two. For row m, the computations follow the order $c_m(m+1), \ldots, c_m(N - M + m)$. Each of these entries is calculated using formula (1). For the valid entries of a row, the total number of comparisons that they are used for computing the next row is $1 + \cdots + N - M + 1$. Hence the complexity for computing row m, $m = 2, \ldots, M$, is of $O(N^2)$, and the overall time complexity is of $O(MN^2)$. The last entry computed, $c_M(N)$, gives the optimal allocation of the N channels to the M users and hence solves 1-MPCA. In parallel, the solution is stored in a second matrix of same size. Solution recording clearly has lower complexity than $O(MN^2)$, and the theorem follows. $\qquad\square$

In the following theorem, we generalize the dynamic programming concept to any positive integer K. The generalized algorithms are able to solve K-MPCA to global optimality.

Theorem 8. *Global optimum of K-MPCA can be computed by dynamic programming in $O(MN^{2K})$ time.*

Proof. Let $N_j = |\mathcal{N}_j|, j = 1, \ldots, K$. For the first m users, denote by $c_m(h_1, \ldots, h_K)$ the optimum power of allocating h_j channels of group, where $h_j \leq N_j$, $j = 1, \ldots, K$. Denote by $p_m^{(k_1, \ldots, k_K)}$ the power for user m, if it is allocated k_j channels of channel group j, $j = 1, \ldots, K$. We introduce the convention that $p_m^{(0, \ldots, 0)} = \infty$ for convenience. By enumerating user m's allocation of channels of the K groups, we obtain the following recursion formula for $c_m(h_1, \ldots, h_K)$.

$$
\begin{aligned}
c_m(h_1, \ldots, h_K) = \min_{k_j \in \{0, \ldots, \min\{h_j, N-M+1\}\}, j=1,\ldots,K} \\
\{p_m^{(k_1, \ldots, k_K)} + c_{m-1}(h_1 - k_1, \ldots, h_K - k_K)\}
\end{aligned}
\tag{2}
$$

Extending the algorithm in the proof of Theorem 7, the corresponding matrix for K-MPCA has dimension $M \prod_{j=1,\ldots,K} N^j$, which does not exceed $O(MN^K)$. To compute an entry, there

11

are no more than $O(N^K)$ calculations (including addition and comparison) using (2). For each calculation, the time required to compute $p_m^{(k_1,\ldots,k_K)}$ is linear[2] in K. Since K is a constant, this computation does not add to the complexity. These observations lead to the overall complexity of $O(MN^{2K})$. Finally, entry $c_M(N^1,\ldots,N^K)$ is clearly the optimum to K-MPCA. The proof is complete by observing that, similar to 1-MPCA, recording the channel allocation solution does not form the computational bottleneck. $\qquad\square$

Remark The polynomial-time tractability of K-MPCA holds only if K is not dependent on M or N. In fact, the general setting of MPCA is equivalent to N-MPCA, i.e., N channel groups with single channel each. The dynamic programming algorithm remains applicable for $K = N$. From Theorem 2, however, the algorithm corresponds to enumerating the solution space, and the running time is exponential. $\square$

Having concluded the tractability of K-MPCA, a natural question to ask next is whether or not the identification of the problem class is tractable as well. Theorem 9 states that this is indeed the case.

Theorem 9. *Recognizing K-MPCA can be performed in $O(MN)$ time for $K = 1$, and in $O(MN^2)$ time for any $K \geq 2$.*

Proof. For $K = 1$, identifying the problem class simply amounts to verifying, for each user, whether or not all the N channels are of the same gain; this immediately leads to the $O(MN)$ time complexity result. For $K \geq 2$, we construct a graph $\mathcal{G}$. The graph has N nodes, each representing a channel in $\mathcal{N}$. Consider two arbitrary channels $n_1, n_2 \in \mathcal{N}, n_1 \neq n_2$. If the two channels have the same gain for each of the users, i.e., $g_{mn_1} = g_{mn_2}, \forall m \in \mathcal{M}$, we denote it by $n_1 \cong n_2$. Checking whether or not this is the case runs obviously in $O(M)$ time. If $n_1 \cong n_2$, we add edge (n_1, n_2) to $\mathcal{G}$. Doing so for all unordered channel pairs has time complexity $O(MN^2)$. Next, note that the equivalence relation of channels is transitive, i.e., if $n_1 \cong n_2$ and

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[2]This result follows by directly applying the single-user rate assignment to K channel groups, where the channels in each group have uniform gain.

$n_2 \cong n_3$, then $n_1 \cong n_3$. Hence channels that are equivalent for all users form a clique in $\mathcal{G}$, whereas channels that differ in gain for at least one user are not connected in $\mathcal{G}$. Consequently the number of strongly connected components in $\mathcal{G}$ equals the number of channel groups, each of which contains channels being equivalent for any user. Identifying the number of strongly connected components requires no more than $O(N^2)$ time for $\mathcal{G}$. Therefore the bottleneck lies in the $O(MN^2)$ complexity of obtaining the graph, and the theorem follows. $\qquad \square$

Remark The tractability results of this section are not restricted to the specific rate/power function defined in the section of system model. The problem class remains tractable (although the overall complexity may grow) for any function in $\mathcal{C}_{\mathrm{inc}}$, as long as the function admits polynomial-time rate allocation of single user. $\square$

6   Conclusions

We have considered the OFDMA resource allocation problem of minimizing the total power of channel allocation, so as to satisfy some rate constraints. Although it has been known that assuming the most general (and ill-behaved) increasing rate functions leads to NP-hard problems, we have shown that the same conclusion holds even if we restrict the class to increasing and concave rate functions. Interestingly, the problem admits a polynomial-time solution if the rate function is an increasing linear function. Hence, progress in the fundamental understanding on the tractability of the problem is made in the following sense: we have sharpened the boundary of tractability to between increasing concave and increasing linear rate functions. Finally, we have also identified specific cases when the problem remains NP-hard, or admits polynomial-time solutions, under various restrictions.

A   Proof of Theorem 1

There is no doubt that MPCA is in NP. The NP-hardness proof uses a polynomial-time reduction from the 3-satisfiability (3-SAT) problem that is NP-complete [16]. A 3-SAT instance consists in a number of boolean variables, and a set of clauses each consisting of a disjunction of exactly three literals. A literal is either a variable or its negation. The output is a yes/no answer to whether or not there is an assignment of boolean values to the variables, such that all the clauses become true. Denote by v and w the numbers of variables and clauses, respectively. For any binary variable z, its negation is denoted by $\hat{z}$. For the proof, we consider 3-SAT where each variable and its negation together appear at most 4 times in the clauses. Note that 3-SAT remains NP-complete with this restriction [17]. Without loss of generality, we assume that each variable z appears in at least one clause, and the same holds for its negation $\hat{z}$, because otherwise the optimal value of the variable becomes known, and the variable can be discarded. Hence the total number of occurrences of each literal in the clauses is between one and three.

We construct an MPCA instance with $M = 2v + w$ and $N = 7v + w$. We categorize the users and channels into groups, and, for convenience, name the groups based on their roles in the proof. The users consist in $2v$ literal users and w clause users. The channels are composed by three groups: v super-channels, $6v$ literal channels, and w auxiliary channels. The rate target $R_m = 1.0, \forall m \in \mathcal{M}$.

Problem reduction is illustrated in Fig. 1. For each binary variable z, three identical literal channels, denoted by z, z' and z'', are defined. A similar construction is done for $\hat{z}$. For this group of six literal channels, one super-channel is defined. We introduce two literal users for the seven channels. One user has channel gain g_l on the three literal channels z, z', and z'', and the other, complementary literal user has channel gain g_l on the remaining three literal channels. Both users have channel gain g_s on the super-channel. See Fig. 1(a). Next, recall that each literal appears at most three times in the clauses in the 3-SAT instance. In the proof, for any binary variable z appearing t times in total in the clauses, with $1 \le t \le 3$, the occurrences are represented by any t elements in $\{z, z', z''\}$ in any order. A similar representation is performed

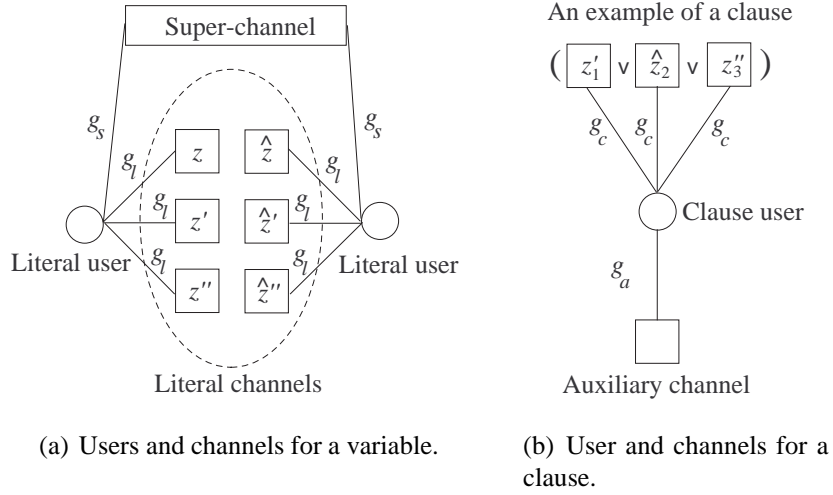(a) Users and channels for a variable.

(b) User and channels for a clause.

Figure 1: An illustration of problem reduction.

for the negation $\hat{z}$. For each clause, we introduce one clause user with gain g_c on the channels corresponding to the original literals in the clause. In addition, one auxiliary channel is defined per clause user with channel gain g_a. See Fig. 1(b). We set $g_s = g_c = 1$, $g_a = \frac{1}{0.9w+0.1}$, $g_l = \frac{g_a}{26} = \frac{1}{26 \cdot (0.9w+0.1)}$. For the user-channel combinations other than those specified, the channel gain is $g_\epsilon = \frac{1}{53w}$. For each user, we refer to the four channels with gain higher than g_ϵ as valid channels, and the other $7v + w - 4$ channels with gain g_ϵ as invalid channels. From the construction, clearly the reduction is polynomial.

We provide several lemmas characterizing the optimum to the MPCA instance. The first three lemmas use the following optimality conditions of single-user rate allocation (e.g., [7]). First, for any user, the derivatives of the power function, evaluated at the allocated rates, are equal on all channels with positive rates. Second, for channels not used, the function derivatives at zero rate are strictly higher than those of the used channels. For $f(x) = \frac{2^x - 1}{g}$, where x is the rate allocated and g is the channel gain, the derivative $f'(x) = \ln(2)\frac{2^x}{g}$.

Lemma 1. *There is an optimum allocation in which no user is allocated any invalid channel.*

Proof. Suppose that at optimum a clause user m_1 is allocated at least one invalid channel. Assume m_1 is also allocated any valid channel, then the invalid channels carry zero rate, because putting the entire rate of 1.0 on the auxiliary channel, the function derivative is at most $\ln(2) \cdot \frac{2}{g_a} = 2\ln(2)(0.9w + 0.1)$, whereas the function derivative for any invalid channel, at zero rate,

15

is $\ln(2) \cdot 53w > 2\ln(2)(0.9w + 0.1)$. Thus the invalid channels can be eliminated from the allocation of m_1. Assume now the auxiliary channel is allocated to another user, say m_2. By construction, the auxiliary channel of m_1 is invalid for m_2. Consider re-allocating any invalid channel of m_1 to m_2, and allocating the auxiliary channel to m_1. Clearly, the total power will not increase. At this stage, the remaining invalid channels allocated to m_1 carry zero flow. Repeating the argument, we obtain an optimal allocation in which no clause user is allocated any invalid channel.

Let m_1 be any literal user and suppose it is allocated one or more invalid channels at optimum. If m_1 is allocated any of its four valid channels, the function derivative at rate 1.0 is at most $\ln(2) \cdot 52(0.9w + 0.1) < \ln(2) \cdot 53w$, and therefore the invalid channels carry zero rate and can be removed from m_1's allocation. Assume therefore all four valid channels are allocated to other users. Consider any literal channel of m_1, and suppose it is allocated to user m_2. For m_2, this literal channel is an invalid one. Swapping the allocation of the literal channel and any invalid channel currently allocated to m_1, the total power will not grow, and the remaining invalid channels allocated to m_1 can be released. The lemma follows from applying the procedure repeatedly. $\qquad\square$

Lemma 2. *If a literal user is allocated its super-channel in the optimal solution, then none of the three literal channels is allocated to the same user.*

Proof. Putting the entire rate of 1.0 on the super-channel, the derivative value is $2\ln(2)$. For any literal channel, the derivative at zero rate is $\frac{\ln(2)}{g_l}$. That $g_l = \frac{1}{26\cdot(0.9w+0.1)}$ and $w \geq 1$ lead to $\frac{\ln(2)}{g_l} > 2\ln(2)$, and the result follows. $\qquad\square$

Lemma 3. *If a literal user is not allocated its super-channel in the optimal solution, then the user is allocated all the three literal channels*

Proof. By the assumption of the lemma and Lemma 1, the literal user in question is allocated one, two, or all three of its literal channels, with total power $f_1 = \frac{1}{g_l}$, $f_2 = 2 \cdot \frac{2^{1/2}-1}{g_l}$, and

16

$f_3 = 3 \cdot \frac{2^{1/3}-1}{g_l}$, respectively. Note that, in the latter two cases, it is optimal to split the rate evenly because of the identical gain values. Clearly, $f_3 < f_2 < f_1$.

We prove that cases one and two are not optimal. Suppose that, at optimum, the literal user is allocated two of the literal channels. Then the remaining literal channel is used to carry a strictly positive amount of flow of a clause user. Consider modifying the solution by allocating all the three literal channels to the literal user, and letting the clause user use the auxiliary channel only. The power saving for the literal user is exactly $f_2 - f_3 > \frac{0.04}{g_l} > \frac{1}{g_a}$, whereas the power increase for the clause user is less than $\frac{1}{g_a}$. This contradicts the optimality assumption. Hence case two is not optimal. Since $f_1 - f_2 > f_2 - f_3$, a similar argument applies to case one, and the result follows. □

By Lemmas 2–3, at optimum, a literal user will use either the super-channel only, or all the three literal channels. Hence, for any literal in the 3-SAT instance, either none or all of the corresponding three literal channels become blocked for the clause users. Consequently, there is a unique mapping between a true/false variable assignment in the 3-SAT instance and the availability of literal channels to the clause users in the MPCA instance. The total power consumption of all the literal users equals exactly $v + 78v \cdot (2^{1/3} - 1)(0.9w + 0.1)$ at optimum. In the remainder of the proof, we concentrate on the power consumption of the clause users.

Lemma 4. *If every clause user is allocated at least one of the three literal channels corresponding to the literals in the clause in the 3-SAT instance, then the total power for all clause users is at most w.*

Proof. Allocating the entire rate of 1.0 to one literal channel gives a power consumption of $\frac{2^1-1}{g_c} = 1$ for the clause user. As there are w clause users, the lemma follows. □

Lemma 5. *If at least one clause user is not allocated any of its three literal channels, the total power for all clause users is strictly higher than w.*

Proof. By the assumption, at least one clause user is allocated the auxiliary channel only, with power $\frac{1}{g_a}$. Each of the other $w - 1$ clause users is allocated at most four channels. Assuming the

availability of all four channels and setting $g_a = g_c$ leads to an under-estimation of the power consumption. The under-estimation has a total power of $4(w-1)\frac{2^{1/4}-1}{g_c} + \frac{1}{g_a} = 4(w-1)(2^{1/4} - 1) + (0.9w + 0.1) > 0.4w - 0.4 + 0.9w + 0.1 = 1.3w - 0.3 \geq w$. □

By Lemmas 4-5, the optimum power for the clause users is at most w if and only if the answer is yes to the 3-SAT instance. Thus the recognition version of MPCA is NP-complete, and its optimization version is NP-hard.

B   Proof of Theorem 4

In the proof of Theorem 1, a 3-SAT instance of v variables and w clauses is reduced to an MPCA instance with $2v + w$ users and $7v + w$ channels, such that no user will be allocated more than three channels at optimum. We make an augmentation by adding $2v$ channels, which we refer to as dummy channels. The channel gain of the dummy channels equals g_ϵ (defined in the proof of Theorem 1) for all users. After the augmentation, there is a total of $9v + w$ channels, organized in three blocks. The sequence of channels is as follows. The first block has $3v$ channels, including the v super-channels and the $2v$ dummy channels, in a sequence of v chunks of 3 channels each. Each chunk is composed by one super-channel and two dummy channels. The next block has the $6v$ literal channels, with v chunks having 6 channels each. Every chunk corresponds to a binary variable z in the 3-SAT instance, and the six literal channels appear in the order $z, z', z'', \hat{z}, \hat{z}', \hat{z}''$. The third block contains the w auxiliary channels. Consider the resulting MPCA instance with the restriction that, for the literal and clause users, respectively, the numbers of channels allocated per user are three and one. In addition, channel allocation must be consecutive in the given sequence.

To prove the hardness result, consider first a relaxation of the problem, in which the two restrictions of channel allocation are ignored for the literal users. For this relaxation, it is clear that Lemmas 1-3 remain valid. By Lemma 1 and the signal-channel restriction of the clause users, each of these users will be allocated one of the four valid channels. Obviously, the

optimum is to allocate one literal channel, or the auxiliary channel if all the three literal channels are allocated to literal users. Thus, as long as at least one literal channel is available to every clause user, the result of Lemma 4 holds. In addition, the validity of Lemma 5 obviously remains. Therefore the optimum to the problem relaxation provides the correct answer to the 3-SAT instance.

By Lemma 3, in the optimum of the relaxed problem, each literal user is either allocated its three consecutive literal channels, or the super-channel. In the latter case, we modify the solution by allocating the two dummy channels accompanying the super-channel, without changing the total power. After the modification, the allocation satisfies the cardinality requirement and restriction of using consecutive channels for all literal users, and the theorem follows.

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